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Methods for obtaining symmetrized representations of SU(2) and the rotation group

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Abstract. A simple formula is obtained by which any symmetrized power of $D(j)$ may be expressed in terms of the symmetrized powers of $D(j-\frac{1}{2})$. This formula does not depend explicitly on the representation theory of the symmetric group and it gives a method of building up any symmetrized power in fully reduced form. In particular, recurrence formulae are obtained for the totally symmetrized and totally antisymmetrized powers of $D(j)$, and a formula is given for arbitrary symmetrized powers of $D(1)$. Finally it is proved that $D(j)^n$ contains $D(j-1)^n$, $n \geqslant 2$, as a proper subrepresentation and formulae are obtained for the symmetrized cubes of $D(j)$.

1. Introduction

The main result of this paper is a formula by which any symmetrized power of the irreducible representation $D(j)$, of dimension $(2j+1)$, of SU(2) may be expressed in terms of the symmetrized powers of $D(j-\frac{1}{2})$ (see theorem (3.3)). This recurrence formula leads to a step-up procedure for obtaining any given symmetrized power in completely reduced form. The beauty of this result is that it does not depend explicitly on the representation theory of the symmetric group S_n and it is very simple to apply. In § 5 an explicit formula is given for the symmetrized powers of $D(1)$, which appears to be new.

The problem of finding the symmetrized powers of $D(j)$ arose when methods were being considered for symmetrizing the representations of point groups (Backhouse and Gard 1974). Let P be a double point group, then in certain cases $D(j)\downarrow \mathrm{P}$ is still an irreducible representation of P, and so the task of symmetrizing such representations is solved if we can symmetrize the representations $D(j)$ themselves. However, the present paper has further direct applications, for example to the theory of term analysis in atomic physics as expounded by Lomont (1959), Smith and Wyborne (1967, 1968) and Wyborne (1969, 1970). For in one approach to the problem of forming correctly antisymmetrized n-particle wavefunctions, the space and spin single-particle states are symmetrized independently according to certain related representations of S_n and then combined, taking into account the Pauli principle. We might add that this work can also be considered as a contribution to the theory of plethysms, a concept which has found mounting application in recent years to atomic and nuclear theory. The problem of symmetrizing representations of the rotation group has also been considered by Murnaghan (1972).

In § 2 we introduce our notation and in § 3 we develop the general method for symmetrizing the representations which leads to the main result, theorem (3.3). In § 4, two recurrence relations are obtained for the totally symmetrized nth power of $D(j)$ and also for the totally antisymmetrized nth power of $D(j)$. In one case the j value is stepped down and in the other case the n value. In § 5 formulae are obtained for the symmetrized powers of $D(\frac{1}{2})$ and $D(1)$. In § 6 we prove the result that $D(j-1)^n$ is contained in $D(j)^n$, $n \geqslant 2$, and obtain a step-up formula for $D(j)^n$. We also obtain a complete set of formulae for the reduction of the symmetrized cubes of $D(j)$ and a recurrence relation for $D(j)^4$.

2. Definitions

It is well known that SU(2) is a 2-1 covering group of the rotation group SO(3) and so we denote the representations of SU(2) by $D(j)$, $j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Integer values of j lead to ordinary representations of SO(3) and half-integer values of j lead to projective representations of SO(3). The inner Kronecker product of two representations of SU(2) has the simple reduced form

$$D(j_1)D(j_2) \equiv \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} D(j). \tag{2.1}$$

Now consider the nth inner Kronecker power of $D(j)$, which we denote by $D(j)^n$. The symmetric group S_n acts on the carrier space of this representation by permuting the basis elements of the $D(j)$'s. This action commutes with the action of SU(2) and hence the carrier space splits into a direct sum of $SU(2) \times S_n$ invariant subspaces Ω^ν, where $[\nu]$ is a unitary irreducible representation of S_n corresponding to the partition $(\nu) = (\nu_1, \nu_2, \ldots, \nu_d)$, $\nu_1 \geqslant \nu_2 \geqslant \ldots \geqslant \nu_d > 0$ of the positive integer n. Note that Ω^ν will be empty if the number of rows d of the Young's diagram (YD) is greater than $(2j+1)$. It can be shown that Ω^ν carries a given representation, which is denoted by $D(j) \otimes (\nu)$, d_ν times, where $d_\nu = \dim[\nu]$. Hence there is a direct sum decomposition

$$D(j)^n \equiv \bigoplus_\nu d_\nu[D(j)\otimes(\nu)] \tag{2.2}$$

summed over all the representations $[\nu]$ of S_n. The representation $D(j)\otimes(\nu)$ is called the symmetrized power of $D(j)$ corresponding to the partition (ν) of n. More details of this decomposition for the general linear group may be found in Boerner (1970). The above result is obtained by regarding each matrix representation of SU(2) as a subgroup of the general linear group.

3. General formula for $D(j) \otimes (\nu)$

The method we propose to use is a generalization of the standard method of deriving results like (2.1). Let $\chi(j)$ denote the character of $D(j)$, then for a rotation through an angle θ, $\chi(j)$ takes the value

$$e^{ij\theta} + e^{i(j-1)\theta} + \ldots + e^{-ij\theta}. \tag{3.1}$$

But symmetrizing the character $\chi(j)$ of SU(2) is equivalent to symmetrizing the representation

$$R(\theta) \mapsto e^{ij\theta} \oplus e^{i(j-1)\theta} \oplus \ldots \oplus e^{-ij\theta} \tag{3.2}$$

of $\overline{\mathrm{SO}(2)}$ (the double group of SO(2)). Having carried through the symmetrization, we revert to the idea of characters and choosing the highest positive index j appearing, proceed to strip off terms like (3.1). Clearly if (3.1) does appear in this decomposition then $D(j)$ lies in the corresponding symmetrized power. In order to symmetrize the representation (3.2) of $\overline{\mathrm{SO}(2)}$, we need the following theorem and its corollary which are easily deducible from the Weyl formula (Weyl 1950, p 331) or as a generalization of result III of Littlewood (1950, p 290).

Theorem (3.1). Let $L_1, L_2, \ldots, L_r$ be representations of the same group G. Let (ν) be a partition of n and let $(\nu_1), (\nu_2), \ldots, (\nu_r)$ be partitions of $n_1, n_2, \ldots, n_r$ respectively, where $n = n_1 + n_2 + \ldots + n_r$. Then

$$(L_1 \oplus L_2 \oplus \ldots \oplus L_r) \otimes (\nu)$$

$$\equiv \bigoplus_{\substack{n_1, n_2, \ldots, n_r \\ \nu_1, \nu_2, \ldots, \nu_r}} \sigma(\nu; \nu_1, \nu_2, \ldots, \nu_r)[L_1 \otimes (\nu_1)] \ldots [L_r \otimes (\nu_r)]$$

where the direct sum is taken over all partitions of n as $n = n_1 + n_2 + \ldots + n_r$ and for each such partition $\sigma(\nu; \nu_1, \nu_2, \ldots, \nu_r)$ is the frequency of the representation $[\nu_1] \times [\nu_2] \times \ldots \times [\nu_r]$ in $[\nu] \downarrow S_{n_1} \times S_{n_2} \times \ldots \times S_{n_r}$.

Denote the linear characters of $\overline{\mathrm{SO}(2)}$ by ψ_p so that

$$\psi_p(\theta) = e^{ip\theta} \tag{3.3}$$

(p is integer or half-integer). Clearly $(\psi_p)^n = \psi_{np}$. We have the following result.

Corollary (3.2).

$$(\psi_j \oplus \psi_{j-1} \oplus \ldots \oplus \psi_{-j}) \otimes (\nu)$$

$$\equiv \bigoplus \sigma(\nu; n_1, n_2, \ldots, n_{2j+1})(\psi_j)^{n_1}(\psi_{j-1})^{n_2} \ldots (\psi_{-j})^{n_{2j+1}} \tag{3.4}$$

where the summation is over all partitions of n as $n = n_1 + n_2 + \ldots + n_{2j+1}$.

This follows immediately from theorem (2.1) since if ψ is a linear character

$$\psi^n = \psi \otimes (n). \tag{3.5}$$

To analyse these results still further we need the theory of outer direct products of symmetric group representations. See for example Robinson (1961) or Hamermesh (1964). The outer direct product, denoted by $\odot$, is defined by

$$[\nu_1] \odot [\nu_2] \odot \ldots \odot [\nu_r] = ([\nu_1] \times [\nu_2] \times \ldots \times [\nu_r]) \uparrow S_n \tag{3.6}$$

where we have used the notation of theorem (2.1) and $\uparrow$ denotes induction. By the Frobenius reciprocity theorem

$$[\nu_1] \odot [\nu_2] \odot \ldots \odot [\nu_r] \equiv \bigoplus_{\nu} \sigma(\nu; \nu_1, \nu_2, \ldots, \nu_r)[\nu]. \tag{3.7}$$

It can be shown that the representation $[n_1] \odot [n_2] \odot \ldots \odot [n_{2j+1}]$ decomposes into representations of S_n corresponding to YD with at most $(2j+1)$ rows. Hence

$\sigma(\nu; n_1, n_2, \ldots, n_{2j+1})$ is identically zero if $[\nu]$ corresponds to a YD with more than $(2j+1)$ rows and consequently $D(j) \otimes (\nu)$ is empty. Also, $D(j) \otimes (1^n)$ will vanish unless each n_i takes the value one or zero, so that $D(j) \otimes (1^n)$ is empty if $n > 2j+1$. Yet another standard result is the following

$$(\psi_j \oplus \psi_{j-1} \oplus \ldots \oplus \psi_{-j}) \otimes (n) \equiv \bigoplus (\psi_j)^{n_1}(\psi_{j-1})^{n_2} \ldots (\psi_{-j})^{n_{2j+1}} \tag{3.8}$$

summed over all partitions of n as $n = n_1 + n_2 + \ldots + n_{2j+1}$. This holds because the frequency of $[n]$ in $[n_1] \odot [n_2] \odot \ldots \odot [n_{2j+1}]$ is one for all partitions.

In order to find a step-up procedure for symmetrizing representations we apply theorem (3.1) to $(L \oplus \psi)$, where L is an arbitrary representation and ψ is a linear character:

$$(L \oplus \psi) \otimes (\nu) = \bigoplus \sigma(\nu; \mu, n_2)[L \otimes (\mu)]\psi^{n_2} \tag{3.9}$$

where (μ) is a partition of $n_1 = n - n_2$. Take $(\nu) = (\nu_1, \nu_2, \ldots, \nu_r)$ where

$$\nu_1 + \nu_2 + \ldots + \nu_r = n \qquad \text{and} \qquad \nu_1 \geqslant \nu_2 \geqslant \ldots \geqslant \nu_r > 0.$$

Take $(\mu) = (\mu_1, \mu_2, \ldots, \mu_s)$ where $\mu_1 + \mu_2 + \ldots + \mu_s = n_1$ and $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_s > 0$. From the theory of outer direct products, the decomposition of $[\mu] \odot [n_2]$ only contains representations $[\nu]$ of S_n corresponding to YD with s or $(s+1)$ rows. Hence $s = r$ or $(r-1)$. In fact it is possible to characterize all representations $[\mu]$ which lead to a non-zero $\sigma(\nu; \mu, n_2)$. They satisfy

$$\left.\begin{array}{l} 0 \leqslant \mu_r \leqslant \nu_r \\ \nu_r \leqslant \mu_{r-1} \leqslant \nu_{r-1} \\ \vdots \\ \nu_2 \leqslant \mu_1 \leqslant \nu_1 \end{array}\right\} \tag{3.10}$$

and in all cases $\sigma(\nu; \mu, n_2) = 1$. Hence

$$(L \oplus \psi) \otimes (\nu) \equiv \bigoplus_{(\mu)\,\text{in}\,(3.10)} [L \otimes (\mu)]\psi^{n-\mu_1-\mu_2\ldots-\mu_r}. \tag{3.11}$$

Now we apply this result to symmetrize representations of the form

$$(\psi_j \oplus \psi_{j-1} \oplus \ldots \oplus \psi_{-j}) \tag{3.12}$$

by taking $\psi = \psi_j$ and $L = \psi_{j-1} \oplus \ldots \oplus \psi_{-j}$. But L can also be written as an inner Kronecker product

$$L = \psi_{(-\frac{1}{2})}[\psi_{(j-\frac{1}{2})} \oplus \psi_{(j-\frac{3}{2})} \oplus \ldots \oplus \psi_{(-j+\frac{1}{2})}]. \tag{3.13}$$

Hence

$$L \otimes (\mu) = \psi_{(-\frac{1}{2})}^{(\mu_1+\mu_2+\ldots+\mu_r)}\{[\psi_{(j-\frac{1}{2})} \oplus \ldots \oplus \psi_{(-j+\frac{1}{2})}] \otimes (\mu)\}. \tag{3.14}$$

Substituting in (3.11) we obtain

$$\begin{aligned}(\psi_j \oplus \psi_{j-1} \oplus \ldots \oplus \psi_{-j}) \otimes (\nu) \\ \equiv \bigoplus_{(\mu)\,\text{in}\,(3.10)} \psi_{nj-(\mu_1+\mu_2+\ldots+\mu_r)(j+\frac{1}{2})}[(\psi_{j-\frac{1}{2}} \oplus \ldots \oplus \psi_{-j+\frac{1}{2}}) \otimes (\mu)].\end{aligned} \tag{3.15}$$

Hence we can express the character $\chi(j) \otimes (\nu)$ in terms of the characters $\chi(j-\frac{1}{2}) \otimes (\mu)$ for all (μ) in the range defined by (3.10). Note that the maximum possible index appearing is $J = nj - (\nu_2 + \nu_3 + \ldots + \nu_r)$ and this is generally not attained.

In the above analysis, the symmetry of the problem has been lost by the choice of L and ψ. We could equally well take $\psi = \psi_{-j}$ and

$$L = \psi_j \oplus \psi_{j-1} \oplus \dots \oplus \psi_{-j+1}$$
$$= \psi_{\frac{1}{2}}[\psi_{(j-\frac{1}{2})} \oplus \psi_{(j-\frac{3}{2})} \oplus \dots \oplus \psi_{(-j+\frac{1}{2})}]. \tag{3.16}$$

This leads to the result

$$(\psi_j \oplus \psi_{j-1} \oplus \dots \oplus \psi_{-j}) \otimes (v)$$
$$\equiv \bigoplus_{(\mu)\text{ in }(3.10)} \psi_{-nj+(\mu_1+\mu_2+\dots+\mu_r)(j+\frac{1}{2})}[(\psi_{j-\frac{1}{2}} \oplus \psi_{j-\frac{3}{2}} \oplus \dots \oplus \psi_{-j+\frac{1}{2}}) \otimes (\mu)]. \tag{3.17}$$

Adding (3.15) and (3.17) we obtain a direct sum of terms of the form

$$(\psi_p \oplus \psi_{-p})(\psi_J \oplus \psi_{J-1} \oplus \dots \oplus \psi_{-J}). \tag{3.18}$$

If $J \geqslant p$, this leads to the representation $D(J+p) \oplus D(J-p)$ of SU(2). If $J < p$, this is formally $D(J+p) - D(-J+p-1)$. Both have the same form if we define

$$D(-j) = -D(j-1). \tag{3.19}$$

Hence we may define the operator T by

$$T(p)[D(J)] = D(J+p) \oplus D(J-p). \tag{3.20}$$

This action of T may be extended by linearity to arbitrary representations of SU(2). Now adding (3.15) and (3.17) and using the definition of T we obtain the following theorem.

Theorem (3.3).

$$2D(j) \otimes (v) \equiv \bigoplus_{(\mu)\text{ in }(3.10)} T[nj-(\mu_1+\mu_2+\dots+\mu_r)(j+\tfrac{1}{2})][D(j-\tfrac{1}{2}) \otimes (\mu)]. \tag{3.21}$$

This appears to be a very useful result since it expresses the representation $D(j) \otimes (v)$ directly in terms of the representations $D(j-\frac{1}{2}) \otimes (\mu)$ belonging to lower j values. Hence starting from low values of j, equation (3.21) gives a step-up procedure for obtaining any symmetrized representation in reduced form. It is very simple to use and is completely independent of the representation theory of S_n.

4. Recurrence relations for $D(j) \otimes (n)$ and $D(j) \otimes (1^n)$

If we restrict our attention to the totally symmetrized nth power, $D(j) \otimes (n)$, and the totally antisymmetrized nth power, $D(j) \otimes (1^n)$, we obtain two recurrence relations as follows:

$$(\psi_j \oplus \psi_{j-1} \oplus \dots \oplus \psi_{-j}) \otimes (n) \equiv \bigoplus_{n_1+n_2+\dots+n_{2j+1}=n} (\psi_j)^{n_1}(\psi_{j-1})^{n_2} \dots (\psi_{-j})^{n_{2j+1}}. \tag{4.1}$$

Taking $n_1, n_{2j+1} \geqslant 1$ gives

$$(\psi_j \oplus \psi_{j-1} \oplus \dots \oplus \psi_{-j}) \otimes (n-2) \tag{4.2}$$

since $\psi_j \psi_{-j} = \psi_0$. When $n_1 = 0$ we obtain the term

$$\psi_{-n/2}[(\psi_{j-\frac{1}{2}} \oplus \psi_{j-\frac{3}{2}} \oplus \dots \oplus \psi_{-j+\frac{1}{2}}) \otimes (n)] \tag{4.3}$$

and when $n_{2j+1} = 0$ we obtain the term

$$\psi_{n/2}[(\psi_{j-\frac{1}{2}} \oplus \psi_{j-\frac{3}{2}} \oplus \ldots \oplus \psi_{-j+\frac{1}{2}}) \otimes (n)]. \tag{4.4}$$

We must subtract those representations which have been counted twice, namely those corresponding to $n_1 = n_{2j+1} = 0$. This term is

$$(\psi_{j-1} \oplus \psi_{j-2} \oplus \ldots \oplus \psi_{-j+1}) \otimes (n). \tag{4.5}$$

Using the definition of the operator T given by (3.19) we obtain

$$D(j) \otimes (n) \equiv D(j) \otimes (n-2) \oplus T(n/2)[D(j-\tfrac{1}{2}) \otimes (n)] - D(j-1) \otimes (n) \tag{4.6}$$

where the operator $T(n/2)$ acts on each irreducible constituent of $D(j-\frac{1}{2}) \otimes (n)$.

Equation (4.6) gives a formula for the difference $[D(j) \otimes (n)] - [D(j) \otimes (n-2)]$ in terms of symmetrized powers of lower j value. This is most useful for representations of low dimension and as an example we obtain $D(3/2) \otimes (n)$.

$$\begin{aligned} &D(3/2) \otimes (n) - D(3/2) \otimes (n-2) \\ &\quad \equiv D(3n/2) \oplus D(3n/2-2) \oplus \ldots \oplus \begin{cases} D(n/2) & n \text{ even} \\ D(n/2+1) & n \text{ odd} \end{cases} \\ &\quad \oplus D(n/2) \oplus D(n/2-2) \oplus \ldots \oplus \begin{cases} D(-n/2) & n \text{ even} \\ D(1-n/2) & n \text{ odd} \end{cases} \\ &\quad - D(n/2). \end{aligned}$$

Hence for all $n \geqslant 4$

$$D(3/2) \otimes (n) - D(3/2) \otimes (n-4) \equiv D(3n/2) \oplus \bigoplus_{j=n/2}^{3n/2-2} D(j). \tag{4.7}$$

If we define

$$\left.\begin{aligned} D(j) \otimes (0) &= D(0) \\ D(j) \otimes (-n) &= 0 \end{aligned}\right\} \tag{4.8}$$

then we can read off the values for $n = 2, 3$ as well.

There is an alternative recurrence formula for the totally symmetrized nth power which is more useful for dealing with low values of n. In equation (4.1) take $n_1 = n_{2j+1} = 0$. This gives rise to the term $D(j-1) \otimes (n)$. Taking successively $n_1 > 0$ and $n_{2j+1} > 0$ gives rise to $T(j)[D(j) \otimes (n-1)]$. We must subtract the representation $D(j) \otimes (n-2)$ corresponding to $n_1 > 0$ and $n_{2j+1} > 0$. Hence

$$D(j) \otimes (n) - D(j-1) \otimes (n) \equiv T(j)[D(j) \otimes (n-1)] - D(j) \otimes (n-2). \tag{4.9}$$

Note that (4.9) and (4.6) may be combined to give an alternative formula. Now we make use of (4.9) to obtain expressions for $D(j) \otimes (2)$ and $D(j) \otimes (3)$.

$$D(j) \otimes (2) - D(j-1) \otimes (2) \equiv [D(2j) \oplus D(0)] - D(0).$$

Hence

$$D(j) \otimes (2) \equiv D(2j) \oplus D(2j-2) \oplus \ldots \oplus \begin{cases} D(0) & 2j \text{ even} \\ D(1) & 2j \text{ odd} \end{cases} \tag{4.10}$$

and

$$\begin{aligned} &D(j) \otimes (3) - D(j-1) \otimes (3) \\ &\quad \equiv [D(3j) \oplus D(3j-2) \oplus \ldots \oplus D(j)] \\ &\quad \oplus [D(j) \oplus D(j-2) \oplus \ldots \oplus D(-j)] - D(j). \end{aligned}$$

Hence

$$D(j)\otimes(3)-D(j-2)\otimes(3)\equiv D(3j)\oplus\bigoplus_{m=j}^{3j-2}D(m). \tag{4.11}$$

In a similar way we obtain recurrence relations for the totally antisymmetrized powers. Starting from the equation

$$(\psi_j\oplus\psi_{j-1}\oplus\ldots\oplus\psi_{-j})\otimes(1^n)$$
$$\equiv\bigoplus_{n=n_1+n_2+\ldots+n_{2j+1}}\sigma(1^n;n_1,n_2,\ldots,n_{2j+1})(\psi_j)^{n_1}\ldots(\psi_{-j})^{n_{2j+1}} \tag{4.12}$$

where $\sigma(1^n;n_1,n_2,\ldots,n_{2j+1})$ is zero unless each $n_i=0,1$ $(i=1,2,\ldots,2j+1)$. Hence $n\leqslant 2j+1$ for a non-trivial result. Taking $n_1=n_{2j+1}=1$ gives rise to $D(j-1)\otimes(1^{n-2})$. Taking first $n_1=0$, then $n_{2j+1}=0$, gives rise to $T(n/2)[D(j-\frac{1}{2})\otimes(1^n)]$. We must subtract the overlap $D(j-1)\otimes(1^n)$ corresponding to $n_1=n_{2j+1}=0$. Hence we obtain the formula

$$D(j)\otimes(1^n)\equiv D(j-1)\otimes(1^{n-2})\oplus T(n/2)[D(j-\tfrac{1}{2})\otimes(1^n)]-D(j-1)\otimes(1^n). \tag{4.13}$$

Applying this result to $D(3/2)$ we obtain

$$\left.\begin{aligned}D(3/2)\otimes(1^2)&\equiv D(2)\oplus D(0)\\ D(3/2)\otimes(1^3)&=D(3/2)\end{aligned}\right\} \tag{4.14}$$

We can obtain another formula for $D(j)\otimes(1^n)$ as follows. Taking $n_1=n_{2j+1}=0$ gives $D(j-1)\otimes(1^n)$. Taking $n_1=0$, $n_{2j+1}=1$ and then $n_1=1$, $n_{2j+1}=0$ gives $T(j)[D(j-1)\otimes(1^{n-1})]$. Taking $n_1=n_{2j+1}=1$ gives $D(j-1)\otimes(1^{n-2})$. Hence

$$D(j)\otimes(1^n)\equiv D(j-1)\otimes(1^n)\oplus D(j-1)\otimes(1^{n-2})\oplus T(j)[D(j-1)\otimes(1^{n-1})]. \tag{4.15}$$

Taking $n=2,3$ we obtain

$$D(j)\otimes(1^2)-D(j-1)\otimes(1^2)=D(2j-1) \tag{4.16}$$

and

$$D(j)\otimes(1^3)-D(j-2)\otimes(1^3)\equiv D(3j-3)\oplus D(3j-5)\oplus D(3j-6)\oplus\ldots\oplus D(j-1). \tag{4.17}$$

For completeness we now list some results which link the totally symmetrized and totally antisymmetrized powers (see Murnaghan 1972). Let n, n' be integers, then:

(i) $D(j)\otimes(n)\equiv D(n/2)\otimes(2j)$

(ii) $D(j)\otimes(1^{2n+1})\equiv D(j-n)\otimes(2n+1)$ $\quad(n\leqslant j)$

(iii) $D(j)\otimes(1^n)\equiv D(j)\otimes(1^{n'})$, $\quad(n+n'=2j+1)$.

The first of these is known as Hermite's law of reciprocity and it is illustrated, for example, by (4.7) and (4.11). All these results can be proved directly from (3.4).

5. Formula for $D(1) \otimes (\nu)$

In this section we obtain explicit formulae for $D(\frac{1}{2}) \otimes (\nu)$ and $D(1) \otimes (\nu)$. From (3.4)

$$\begin{aligned}
(\psi_{\frac{1}{2}} \oplus \psi_{-\frac{1}{2}}) \otimes (\nu_1, \nu_2) &\\
&\equiv \bigoplus_{n_1+n_2=n} \sigma(\nu; n_1, n_2)\psi_{\frac{1}{2}(n_1-n_2)} \\
&\equiv \bigoplus_{n_1=\nu_2}^{\nu_1} \psi_{\frac{1}{2}(2n_1-n)} \\
&= \psi_{\frac{1}{2}(\nu_2-\nu_1)} \oplus \psi_{\frac{1}{2}(\nu_2-\nu_1+2)} \oplus \ldots \oplus \psi_{\frac{1}{2}(\nu_1-\nu_2)}.
\end{aligned}$$

Hence

$$D(\tfrac{1}{2}) \otimes (\nu_1, \nu_2) = D[\tfrac{1}{2}(\nu_1 - \nu_2)]. \tag{5.1}$$

This is a well known result, but we have not seen the explicit formula for $D(1) \otimes (\nu)$ which we now obtain. From (3.4)

$$\begin{aligned}
(\psi_1 \oplus \psi_0 \oplus \psi_{-1}) \otimes (\nu_1, \nu_2, \nu_3) &\\
&\equiv \bigoplus \sigma(\nu; \mu, n_2)[(\psi_1 \oplus \psi_{-1}) \otimes (\mu)] \\
&= \bigoplus_{\substack{\nu_2 \leqslant \mu_1 \leqslant \nu_1 \\ \nu_3 \leqslant \mu_2 \leqslant \nu_2}} (\psi_1 \oplus \psi_{-1}) \otimes (\mu_1, \mu_2)
\end{aligned}$$

where $(\mu) = (\mu_1, \mu_2)$ and the summation limits are obtained from (3.10). From the previous case and (3.2) we obtain

$$(\psi_1 \oplus \psi_{-1}) \otimes (\mu_1, \mu_2) \equiv \psi_{(\mu_1-\mu_2)} \oplus \psi_{(\mu_1-\mu_2-2)} \oplus \ldots \oplus \psi_{(-\mu_1+\mu_2)} \tag{5.2}$$

giving alternate values of the index j.

Consider all the allowed pairs (μ_1, μ_2) defined by (3.10) and plot these points on a graph. They lie inside a rectangle with sides $\mu_1 = \nu_1$, $\mu_2 = \nu_2$, $\mu_1 = \nu_2$, $\mu_2 = \nu_3$. From (5.2), adjacent pairs of points in either the μ_1 or μ_2 directions lead to a representation of SU(2), except for points on the diagonal $\mu_1 = \mu_2$ which may lead to $D(0)$. In particular we have

$$D(1) \otimes (n) \equiv D(n) \oplus D(n-2) \oplus \ldots \oplus \begin{cases} D(0) & n \text{ even} \\ D(1) & n \text{ odd.} \end{cases} \tag{5.3}$$

For an arbitrary partition (ν), we analyse the diagram first by considering the trapezium bounded by the lines $\mu_1 = \nu_2$, $\mu_1 = \mu_2$, $\mu_1 = \nu_1$, $\mu_2 = \nu_3$ and then removing the triangle bounded by the sides $\mu_1 = \mu_2$, $\mu_1 = \nu_1$, $\mu_2 = \nu_2 + 1$. The contribution to the representation from each area can be written down using the notation of (5.3) and we obtain

$$\begin{aligned}
D(1) \otimes (\nu_1, \nu_2, \nu_3) &\\
&\equiv [D(1) \otimes (\nu_1 - \nu_3)] \oplus [D(1) \otimes (\nu_1 - \nu_3 - 1)] \\
&\quad \oplus \ldots \oplus [D(1) \otimes (\nu_2 - \nu_3)] - \{[D(1) \otimes (\nu_1 - \nu_2 - 1)] \\
&\quad \oplus [D(1) \otimes (\nu_1 - \nu_2 - 2)] \oplus \ldots \oplus D(1) \oplus D(0)\}.
\end{aligned} \tag{5.4}$$

(5.4) may be expanded using (5.3). We may also take dots in pairs along the μ_1 direction which leads to the result

$$
\begin{aligned}
D(1) &\otimes (\nu_1, \nu_2, \nu_3) \\
&\equiv [D(1) \otimes (\nu_1 - \nu_3) \oplus D(1) \otimes (\nu_1 - \nu_3 - 1) \oplus \ldots \oplus D(1) \otimes (\nu_1 - \nu_2)] \\
&\quad - [D(1) \otimes (\nu_2 - \nu_3 - 1) \oplus D(1) \otimes (\nu_2 - \nu_3 - 2) \oplus \ldots \oplus D(1) \oplus D(0)]. \qquad (5.5)
\end{aligned}
$$

The formulae (5.4) and (5.5) contain too much information and so it is more practical to use the step-up formula

$$
\begin{aligned}
D(1) &\otimes (\nu_1 + 2, \nu_2, \nu_3) \\
&\equiv D(1) \otimes (\nu_1, \nu_2, \nu_3) \oplus D(\nu_1 - \nu_3 + 2) \\
&\quad \oplus D(\nu_1 - \nu_3 + 1) \oplus \ldots \oplus D(\nu_1 - \nu_2 + 2) \qquad (5.6)
\end{aligned}
$$

in conjunction with the results

$$D(1) \otimes (\nu_1, \nu_2, \nu_3) \equiv D(1) \otimes (\nu_1 - \nu_3, \nu_2 - \nu_3) \qquad (5.7)$$

and

$$
\left.
\begin{aligned}
&D(1) \otimes (\nu_2, \nu_2) \equiv D(1) \otimes (\nu_2) \\
&D(1) \otimes (\nu_2 + 1, \nu_2) \equiv D(\nu_2 + 1) \oplus D(\nu_2) \oplus \ldots \oplus D(1)
\end{aligned}
\right\}. \qquad (5.8)
$$

The result (5.7) is to be found in Hamermesh (1964, p 391) and (5.8) follows from (5.4).

6. Analysis of Kronecker powers

The main purpose of this section is to prove the result that $D(j)^n$ contains $D(j-1)^n$ as a proper subrepresentation and also to suggest an alternative approach to symmetrized powers. As in § 3 we consider the character of $D(j)^n$ to be associated with the representation

$$(\psi_j \oplus \psi_{j-1} \oplus \ldots \oplus \psi_{-j})^n \qquad (6.1)$$

of $\overline{\mathrm{SO}(2)}$. The terms in the decomposition of this representation are in one-to-one correspondence with the n-tuples $(j_1, j_2, \ldots, j_n)$ where $j_r = -j, \ -j+1, \ldots, j$ and $r = 1, 2, \ldots, n$. Each n-tuple $(j_1, j_2, \ldots, j_n)$ is associated with the representation $\psi_{j_1 + j_2 + \ldots + j_n}$ of $\overline{\mathrm{SO}(2)}$. Hence we have an n-dimensional simple-cubic lattice structure, each point of which corresponds to a representation of $\overline{\mathrm{SO}(2)}$. We obtain the representation of SU(2) by systematically removing points corresponding to

$$(\psi_J \oplus \psi_{J-1} \oplus \ldots \oplus \psi_{-J}),$$

where J is the largest index appearing, and recording the value of J.

Theorem (*6.1*). If n is any positive integer, $n \geqslant 2$, then $D(j)^n - D(j-1)^n$ is a proper representation of SU(2).

Proof. For $n \geqslant 4$ we may proceed by induction:

$$D(j)^n - D(j-1)^n = D(j)^2[D(j)^{n-2} - D(j-1)^{n-2}] \oplus [D(j)^2 - D(j-1)^2]D(j-1)^{n-2}. \qquad (6.2)$$

Hence the result will be true for all n if we can prove it true for $n = 2, 3$. From (2.1)

$$D(j)^2 \equiv \bigoplus_{m=0}^{2j} D(m). \tag{6.3}$$

So

$$D(j)^2 - D(j-1)^2 \equiv D(2j) \oplus D(2j-1) \tag{6.4}$$

as required.

To find $D(j)^3$ we return to the idea of a simple-cubic lattice. Each triplet (j_1, j_2, j_3) corresponds to a representation $\psi_{j_1+j_2+j_3}$ of $\overline{\mathrm{SO}(2)}$. The representation $D(j)^3 - D(j-1)^3$ may be associated with the triplets on the surface of the cube, ie those with at least one entry j or $-j$. The surface of the cube has six faces and we take adjacent faces in pairs so that each pair contains both the point (j, j, j) and $(-j, -j, -j)$. The contribution to the representation of SU(2) is obtained by connecting adjacent points $-J, -J+1, \ldots, J$ systematically in L-shaped patterns. The contribution from each pair of faces is

$$D(3j) \oplus D(3j-1) \oplus \ldots \oplus D(j). \tag{6.5}$$

The overlap between the sides of the faces gives

$$2D(3j) \oplus D(3j-1). \tag{6.6}$$

Hence

$$D(j)^3 - D(j-1)^3 \equiv D(3j) \oplus 2D(3j-1) \oplus 3[\mathrm{D}(3j-2) \oplus \ldots \oplus D(j)] \tag{6.7}$$

and the theorem follows by induction.

Note that by combining equations (6.7), (4.11) and (4.17) we obtain the formula

$$\begin{aligned} D(j) \otimes (2, 1) - D(j-2) \oplus (2, 1) &\\ \equiv D(3j-1) \oplus D(3j-2) \oplus D(3j-3) &\\ \oplus 2[D(3j-4) \oplus \ldots \oplus D(j)] \oplus D(j-1). & \end{aligned} \tag{6.8}$$

In the light of theorem (6.1) there are now two methods of analysing $D(j)^4 - D(j-1)^4$. Either use (6.2) or else work out the contribution from the surface of the hypercube of lattice points in four-dimensional space. The following result is obtained:

$$\begin{aligned} D(j)^4 - D(j-1)^4 &\\ \equiv D(4j) \oplus 3D(4j-1) \oplus 6D(4j-2) \oplus 10D(4j-3) \oplus 14D(4j-4) &\\ \oplus \ldots \oplus (8j-2)D(2j) \oplus (8j-2)D(2j-1) \oplus (8j-6)D(2j-2) &\\ \oplus \ldots \oplus 10D(2) \oplus 6D(1) \oplus 2D(0). & \end{aligned} \tag{6.9}$$

This obviously gives a quick method of building up powers and it can also be used to find symmetrized powers if we can identify the correct parts of the hypercube. For instance, from (3.8), $D(j) \otimes (n)$ can be identified with n-tuples $(j_1, j_2, \ldots, j_n)$ satisfying $j_1 \geqslant j_2 \geqslant \ldots \geqslant j_n$. Also $D(j) \otimes (l^n)$ can be identified with n-tuples satisfying

$$j_1 < j_2 < \ldots < j_n.$$

These relations can be used to obtain an alternative proof of (4.9) and (4.15). Hence if we could find conditions on the n-tuples corresponding to $D(j) \otimes (\nu)$, restricting them to a given subset of the hypercube, this would give an alternative method of obtaining the required decomposition.

We can also obtain a formula, analogous to (3.21), for $D(j)^n$. From (6.1) it follows that

$$D(j)^n = \left[T(j)+T(j-1)+\ldots+\begin{Bmatrix}T(0)\\T(\frac{1}{2})\end{Bmatrix}\right]D(j)^{n-1} \tag{6.10}$$

where the last term is $T(0)$ or $T(\frac{1}{2})$ corresponding to j being integer or half-integer. This is equivalent to the result obtained by Murnaghan (1972).

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